Linear systems for gravitational-wave detectors

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Section 1

Preliminaries

Bilateral Laplace transform

Given a time-domain function $g(t)$, its Laplace transform $\tilde{g}(s)$ is

$$\tilde{g}(s) = \int_{-\infty}^{+\infty} \mathrm{d}t \, \mathrm{e}^{-st} \, g(t)$$

The variable s is a *complex angular frequency.* We'll also write the Laplace transform as an operator $g(t) \xrightarrow{\mathcal{L}} \tilde{g}(s)$.

When $s = \mathrm{i}\Omega$, this is also a Fourier transform: $g(t) \xrightarrow{\mathcal{F}} \tilde{g}(\Omega)$.

Some Laplace transform properties

Time shifts become phase delays: $g(t-\tau) \xrightarrow{\mathcal{L}} \mathrm{e}^{-s\tau}\tilde{g}(s)$

Time derivatives become frequency rescalings: $\dot{g}(t) \xrightarrow{\mathcal{L}} s\,\tilde{g}(s)$

In particular:

- $\dot{x}(t) \xrightarrow{\mathcal{L}} s\,\tilde{x}(s) = -\mathrm{i}\Omega\tilde{x}(\Omega)$
- $\ddot{x}(t) \xrightarrow{\mathcal{L}} s^2\tilde{x}(s) = -\Omega^2\tilde{x}(\Omega)$

Solving equations in the Laplace domain

Damped harmonic oscillator (time domain):

$$m\ddot{x}(t) = -m\Omega_0^2 x(t) - m\Gamma\dot{x}(t) + F(t)$$

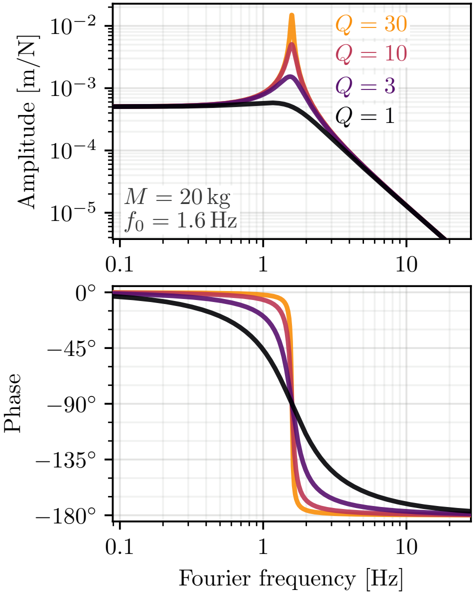
Damped harmonic oscillator (Laplace domain):

$$ms^2\tilde{x}(s) = -m\Omega_0^2\tilde{x}(s) - m\Gamma s\,\tilde{x}(s) + \tilde{F}(s)$$

Allows us to write the system's *frequency response* or *transfer function*:

$$\frac{\tilde{x}(s)}{\tilde{F}(s)} = \frac{1/m}{\Omega_0^2 + s^2 + \Gamma s}$$

Frequency response of the damped harmonic oscillator



Taking $s = \mathrm{i}\Omega$ and defining $Q = \Omega_0/\Gamma$:

$$\frac{\tilde{x}(\Omega)}{\tilde{F}(\Omega)} = \frac{1/m}{\Omega_0^2 - \Omega^2 + \mathrm{i}\Omega_0\Omega/Q}.$$

Zero–pole–gain representation

If $\tilde{g}(s)$ is rational, it can be factorised:

$$\tilde{g}(s) = k\frac{(s - z_0)(s - z_1)(s - z_2)\cdots}{(s - p_0)(s - p_1)(s - p_2)\cdots}$$

where $\{z_i\}$ are the zeros, $\{p_i\}$ the poles, and k the gain.

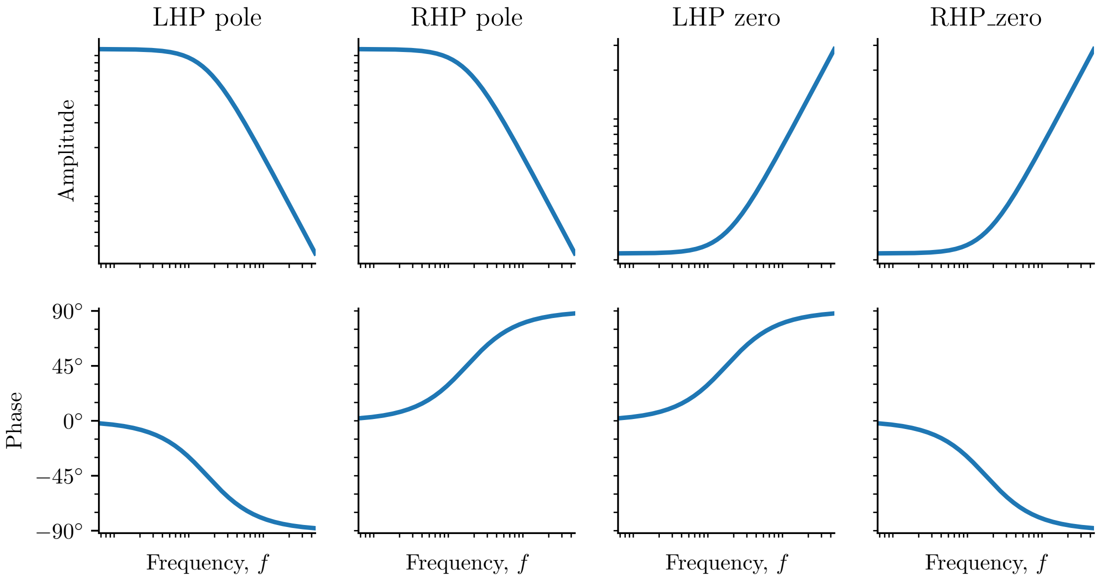
Also, if $\tilde{g}(0)$ is finite and nonzero, we usually prefer to write

$$\tilde{g}(s) = \tilde{g}(0)\frac{(1 - s/z_0)(1 - s/z_1)(1 - s/z_2)\cdots}{(1 - s/p_0)(1 - s/p_1)(1 - s/p_2)\cdots}$$

where

$$\tilde{g}(0) = k \times \frac{\prod_i p_i}{\prod_i z_i}$$

Taxonomy of poles and zeros



For the damped harmonic oscillator:

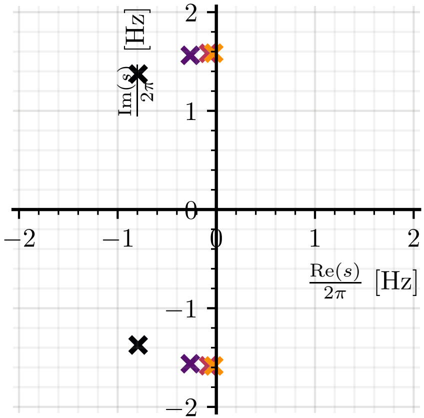
$$\frac{\tilde{x}(s)}{\tilde{F}(s)} = \frac{1/m}{(s - p_+)(s - p_-)}$$

where $p_\pm = -\frac{\Omega_0}{2Q}\left(1 \pm \mathrm{i}\sqrt{4Q^2 - 1}\right)$.

Note

- $|p_\pm| = \Omega_0$
- $\arg p_\pm = \arccos(1/2Q)$.

For $Q \gg 1$, $p_\pm \approx -\Omega_0(1/2Q \pm \mathrm{i})$.



Exercise: Bode plots of some LTI systems

By hand, sketch a Bode plot (i.e., the amplitude and phase) of the frequency response for some or all of the following cases:

1. A real pole at $s = -2\pi \times 10\,\mathrm{Hz}$ and a real zero at $s = -2\pi \times 100\,\mathrm{Hz}$, with a gain of 1 at ac ($f = \infty\,\mathrm{Hz}$).
2. A real pole at $s = -2\pi \times 100\,\mathrm{Hz}$ and a real zero at $s = +2\pi \times 100\,\mathrm{Hz}$, with a gain of 1 at dc ($f = 0\,\mathrm{Hz}$).
3. (Challenge) A complex pair of poles at $s = 2\pi \times (-0.1\,\mathrm{Hz} \pm 10\mathrm{i}\,\mathrm{Hz})$, a real pole at $s = 2\pi \times 0\,\mathrm{Hz}$, and a real zero at $s = -2\pi \times 10\,\mathrm{Hz}$, with a gain of 1 at 100 Hz.

Draw your Bode plots with the horizontal axis in ordinary frequency $f = \Omega/2\pi = s/2\pi\mathrm{i}$.

Without a calculator, estimate the phase of the frequency responses at $f = 10\,\mathrm{Hz}$, $f = 100\,\mathrm{Hz}$, and $f = 1000\,\mathrm{Hz}$.

Measuring the frequency response

A frequency response is also known as a **transfer function** or **input–output relation** or **scattering matrix**.

Taking the example $\tilde{x}(f)/\tilde{F}(f)$, the transfer function is measured at each frequency $f_j = \Omega_j/2\pi$ by:

- Injecting a sine wave $F_j \sin(\Omega_j t)$
- Recording the sinusoidal response $x_j \sin(\Omega_j t + \phi_j) + n(t)$, which has random noise n

The transfer function at that frequency is the complex number $(x_j/F_j)\mathrm{e}^{-\mathrm{i}\phi_j}$.

Linear, time-invariant systems (LTI)

It turns out the systems we have dealt with so far are **linear** (two solutions can always be superposed to make a third valid solution) and **time invariant** (time delaying the input simply time delays the output).

Any such system is complete characterized by its frequency response, and has the property that a sine wave at the input always produces a sine wave at the output, although it may be phase delayed.

Impulse response

If $\tilde{H}(s)$ is a frequency response, then the inverse Laplace transform $H(t) \xleftarrow{\mathcal{L}^{-1}} \tilde{H}(s)$ defines the **impulse response** of the system.

The name is literal: it is the system's time-domain response to a quick, pulse-like force.

Bruel & Kjær impact hammer

Stability: poles in the left or right half-plane

A system is **stable** if and only if **all poles in the system have negative real part** ("left half-plane"). A perturbation to the system decays to zero as $t \to \infty$.

If **any pole has a positive real part** ("right half-plane"), the impulse response diverges as $t \to \infty$. The system is **unstable**.