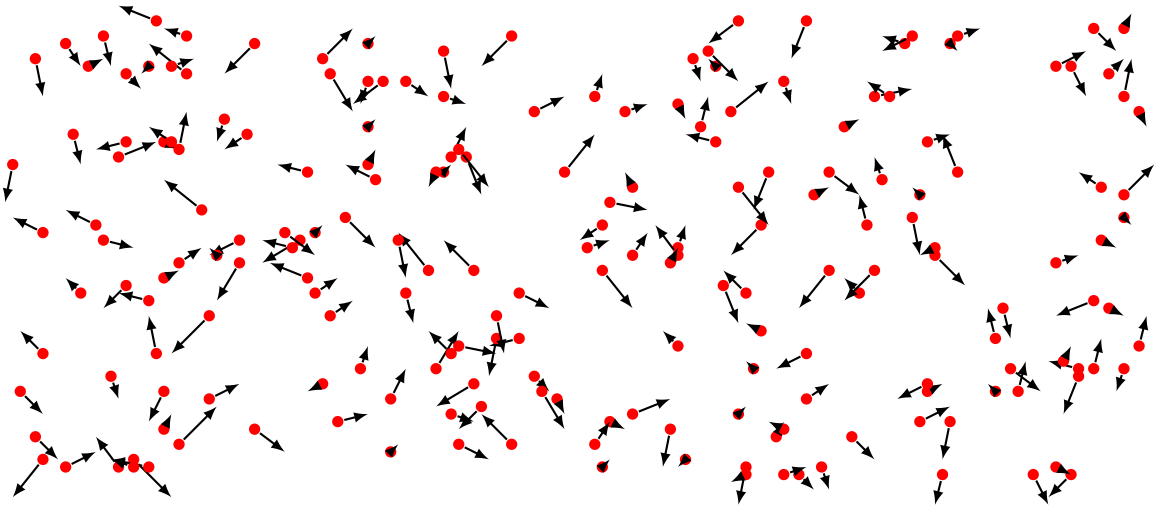
Section 2

# Signal and noise

# Randomness

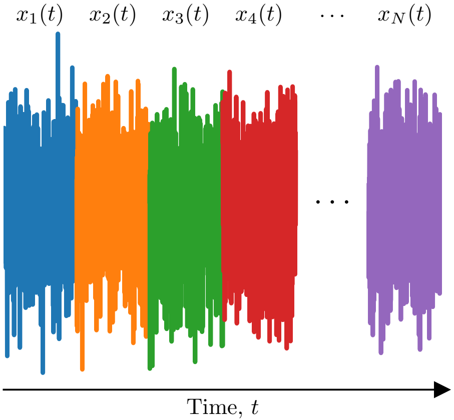


# Random noise and the ensemble average

Consider a (long) noise time series $x(t)$.

We can divide it into $N$ segments: $x_1(t), x_2(t), \ldots, x_N(t)$.

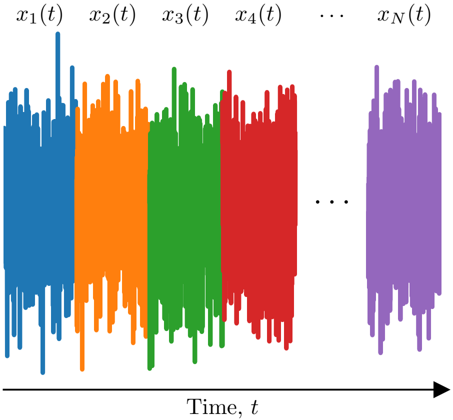
The collection of these segments is an **ensemble** of noise realizations.



# The ensemble average

Noise is random, but its **ensemble-averaged properties** are deterministic.

For example, the mean value of each realization $\overline{x}_1, \overline{x}_2, \dots, \overline{x}_N$ is random. However, its ensemble average $\langle \overline{x} \rangle$ has a single, well-defined value.



# Power spectral density

Given an ensemble of time series $x_1(t), x_2(t), \ldots, x_N(t)$, each of length $T$, the *power spectral density* is the ensemble average of the normalized square magnitude of the Fourier transforms:

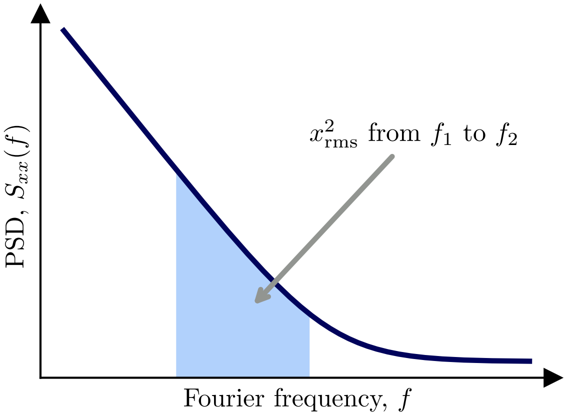
$$S_{xx}(f) = \frac{2}{T} \left\langle |\tilde{x}(f)|^2 \right\rangle ,$$

where $T$ is understood to be arbitrarily large. (The factor of 2 will be explained later.)

The quantity $\sqrt{S_{xx}(\Omega)}$ is called the *amplitude spectral density*.

# Why power spectral density?

Most random noises have fluctuations at all frequency scales. Power spectral density tells us how to compute the expected mean-square noise power occurring between any two frequencies:



# Single- and double-sided densities

In classical mechanics, $x(t)$ is real. Consequently, $\tilde{x}(f)$ satisfies $\tilde{x}(-f) = \tilde{x}^*(+f)$.

Therefore, negative frequencies are redundant. However, they still contain half of the total noise power.

Experimentalists usually work with a single-sided spectral density $S(f)$, defined for $f \in [0, \infty)$, and a factor of 2 is included to account for the power in negative frequencies.

Theorists sometimes work with the double-sided spectral density, defined for $f \in (-\infty, \infty)$.

Unfortunately, papers are not always explicit about which convention they use!

# Two spectral densities to commit to memory

The voltage fluctuations across ideal resistor with resistance $R$ at temperature $T$ has single-sided power spectral density

$$S_{VV}(f) = 4k_{\mathrm{B}}TR \qquad (k_{\mathrm{B}} = \text{Boltzmann's constant}).$$

The optical power fluctuations due to photon shot noise in a beam of light of frequency $\nu$ and power $P$ has single-sided power spectral density

$$S_{PP}(f) = 2h\nu P \qquad (h = \text{Planck's constant}).$$

# Cross-spectral density

Given two random noises $x$ and $y$, their **cross-spectral density** is

$$S_{xy}(f) = \frac{2}{T}\langle \tilde{x}^*(f)\tilde{y}(f)\rangle.$$

The cross-spectral density in general is complex-valued.

# Coherence

For random noises $x$ and $y$, their **magnitude-squared coherence** (often just **coherence**) is

$$|\gamma|^2 = \frac{|S_{xy}(f)|^2}{S_{xx}(f)S_{yy}(f)}.$$

By the Cauchy–Schwartz inequality, $0 \leq |\gamma|^2 \leq 1$.

When $|\gamma|^2 = 0$, the two noises $x$ and $y$ are completely uncorrelated. When $|\gamma|^2 = 1$, the two noises are completely correlated.

Pay attention to how the ensemble averaging is done:

$$|\gamma|^2 = \frac{|\langle \tilde{x}^*(f)\tilde{y}(f)\rangle|^2}{\langle \tilde{x}^*(f)\tilde{x}(f)\rangle\langle \tilde{y}^*(f)\tilde{y}(f)\rangle}.$$

# Discrete Fourier transform

Data records these days are discrete. So is the Fourier transform:

$$\tilde{x}(f_j) = \sum_{i=0}^{N-1} x(t_k) \exp\left(-2\pi \mathrm{i} f_j \, k \, \Delta t\right).$$

(This follows the `numpy` conventions.)

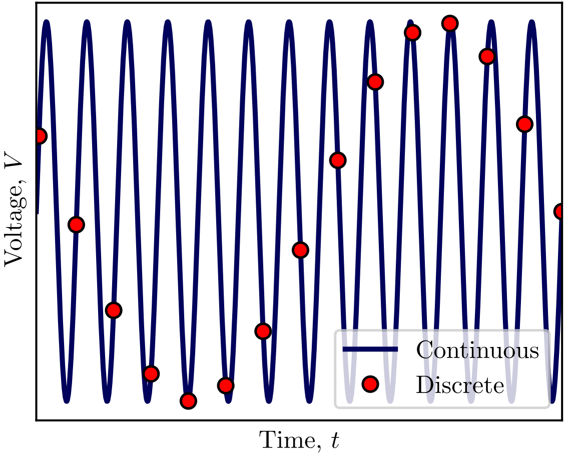
Note that $N = T\,\Delta t$, and that

- Frequency points are separated by $\Delta f = 1/T$ (the sampling frequency).
- The highest frequency is $f_{\mathrm{Nyq}} \leq 1/(2\Delta t)$ (the *Nyquist frequency*).

# Complications of discretisation: aliasing

Signal content above the Nyquist frequency is aliased below the Nyquist frequency.

The analog signal must be filtered (anti-aliased) before being digitised to remove this frequency content.

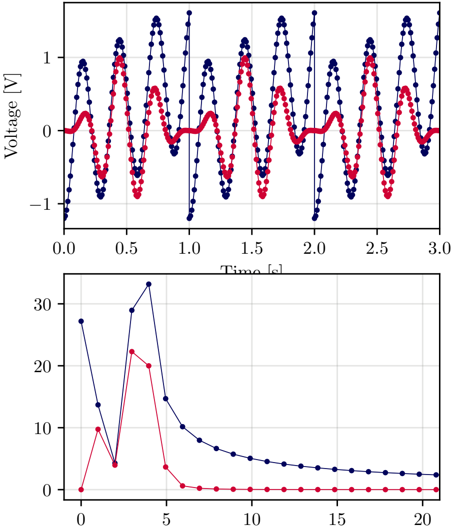


# Complications of discretisation: periodicity

The DFT operates on a periodic domain.

Discontinuities at the ends of a finite time series appear as (usually unwanted) features in the DFT. (Generically, step-like functions in the time domain produce Laplace-domain functions like $1/s$.)

Most spectral density estimation algorithms **window** or **taper** the time series to zero at either end to reduce this effect. See `scipy.signal.windows` for some common ones.



# Implementation

The “standard” method of Welch’s overlapping periodograms:

```
freq_arr, csd_arr = scipy.signal.csd(
    x, # Time series x(t)
    y, # Time series y(t)
    fs=<val>, # Sampling frequency of time series (in hertz)
    window="hann", # Hann is a nice general-purpose window function
    nperseg=<val>, # Number of points in each segment
    noverlap=<val>, # Defaults to nperseg // 2, good choice for Hann window
    nfft=None, # Zero-pads data. Never use this: leave it as None.
    detrend="linear", # Usually better than "constant"
    return_onesided=True, # The single-sided (experimentalist) convention
    scaling="density", # Scales to spectral density units
    average="mean",
    )
```

Section 3

# Feedback control

# Feedback on a harmonic oscillator

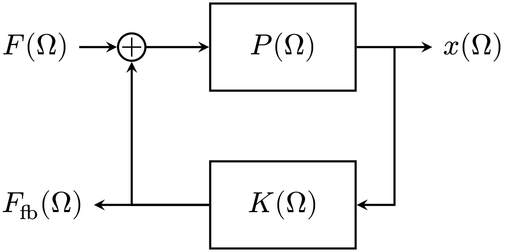
Let's return to the harmonic oscillator, which can be written very simply as

$$x(\Omega) = P(\Omega)F(\Omega),$$

where we have now collected the oscillator's force-to-displacement frequency response into the **plant** $P(\Omega)$.

Let's suppose we can devise a force $F_{\text{fb}}(\Omega) = K(\Omega)x(\Omega)$ that is a filtered version of the oscillator displacement, and apply it to the oscillator.

We call $K(\Omega)$ the **controller**.



# Control alters the system dynamics

Feedback alters our oscillator's frequency-domain equation to

$$x(\Omega) = P(\Omega)[F(\Omega) + F_{\mathrm{fb}}(\Omega)] = P(\Omega)[F(\Omega) + K(\Omega)x(\Omega)].$$

Solving for $x$,

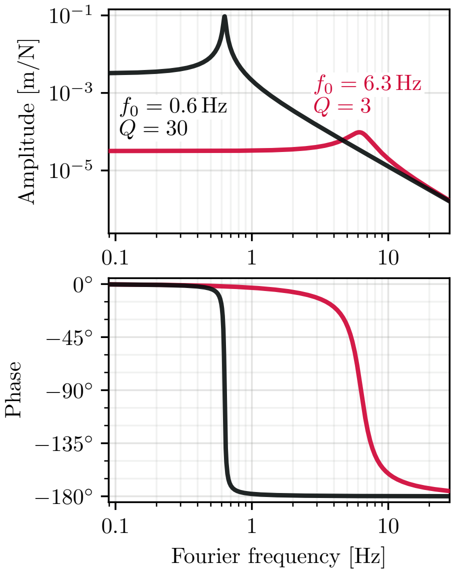
$$x(\Omega) = \frac{P(\Omega)}{1 - P(\Omega)K(\Omega)} F(\Omega).$$

We find that the system's bare plant $P$ has been altered to $P/(1 - PK)$.

# A proportional–velocity controller

Let's choose $F_{\text{fb}}(\Omega) = (-m\Omega_{\text{fb}}^2 + \mathrm{i}m\Gamma_{\text{fb}}\Omega)x(\Omega)$. Then

$$\frac{P}{1-PK} = \frac{1/m}{\Omega_0^2 + \Omega_{\text{fb}}^2 - \Omega^2 + \mathrm{i}(\Gamma + \Gamma_{\text{fb}})\Omega}.$$



# Loop gains

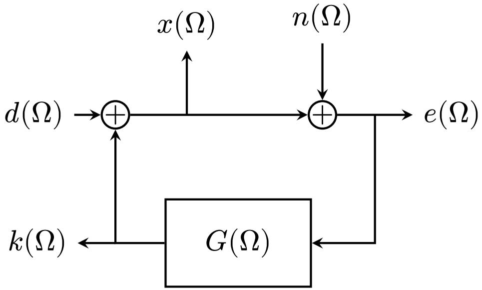
The dimensionless frequency response $G(s) = P(s)K(s)$ is the **open-loop gain**.

Two closely related quantities

$$\frac{1}{1 - G(s)} \qquad \text{and} \qquad \frac{G(s)}{1 - G(s)}$$

are both called **closed loop gain** in various contexts (also "sensitivity" and "complementary sensitivity", which we will not use).

# A canonical control loop with noise




The error signal is given by

$$e(\Omega) = \frac{1}{1 - G(\Omega)} [d(\Omega) + n(\Omega)]$$

When the loop gain is large, $e(\Omega) \simeq -[d(\Omega) + n(\Omega)]/G(\Omega)$: the external disturbance is suppressed.

The control signal is given by

$$k(\Omega) = \frac{G(\Omega)}{1 - G(\Omega)} [d(\Omega) + n(\Omega)]$$

When the loop gain is large, $k(\Omega) \simeq -[d(\Omega) + n(\Omega)]$: the control signal cancels the external disturbance.

# Loop stability

The frequency response of a feedback loop to external disturbances depends fundamentally on $1/(1-G)$.

Like any other linear system, this frequency response is only stable if the poles of $1/(1-G)$ are all in the left-hand plane.

Alternately, one can see by inspection that a sufficient (but not necessary) criterion for instability is the **Bode criterion**: if $G(s) = +1$ for some frequency $s$, then a finite disturbance at that frequency produces an infinite response at that frequency.

Note, this is a complex-valued criterion: instability occurs only if $|G(s)| = 1$ and $\arg G(s) = 0$.

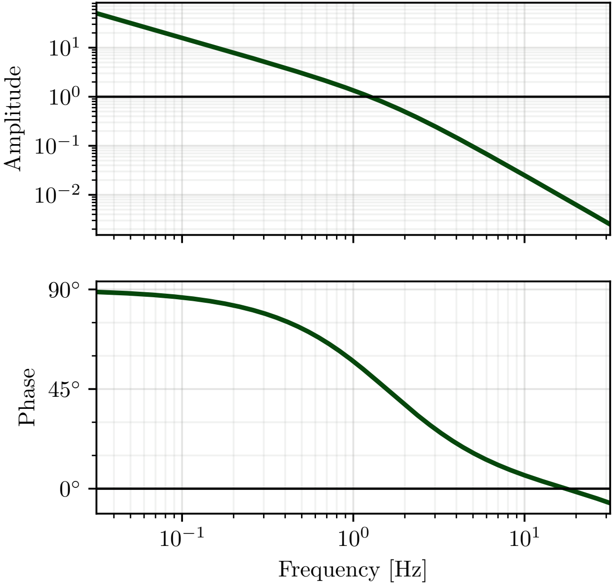
# Unity gain frequency, and gain and phase margin

The frequency $f_\star$ for which $|G(f_\star)| = 1$ is the **unity-gain frequency**.

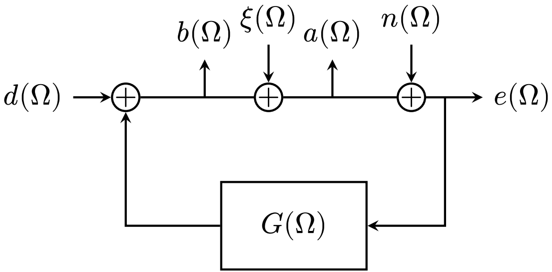
The phase $\arg G(f_\star)$ at the unity gain frequency is the loop's **phase margin**. It characterizes how much additional phase delay could be added to the loop before it becomes unstable.

For any $f$ such that $\arg G(f) = 0$, the magnitude $|G(f)|$ constitute a **gain margin** for the loop. It characterizes how much the gain magnitude can be changed before the loop becomes unstable.

# An example open-loop gain



# Transfer function measurement




By injecting an excitation $\xi(\Omega)$ and measuring before $[b(\Omega)]$ and after $[a(\Omega)]$, we can estimate $G(\Omega)$:

$$\begin{aligned} \hat{G} &= \frac{\langle \xi^* b \rangle}{\langle \xi^* a \rangle} \\ &= \frac{G\langle \xi^* \xi \rangle + \langle \xi^* d \rangle + G\langle \xi^* n \rangle}{\langle \xi^* \xi \rangle + \langle \xi^* d \rangle + G\langle \xi^* n \rangle} \\ &\simeq G \end{aligned}$$

if $\xi$ is sufficiently strong.